YOUNG BASIS, WICK FORMULA, AND HIGHER CAPELLI IDENTITIES

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Abstract. We prove Capelli type identities which involve the whole universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(n))$ and matrix elements of irreducible representations of the symmetric group. These identities generalize higher Capelli identities for the center of $\mathcal{U}(\mathfrak{gl}(n))$ introduced in the author's paper [Ok]. The main role in the proof play the Jucis-Murphy elements.

1. Introduction

1.1. Identify the the standard generators E_{ij} of the Lie algebra with the following vector fields on the vector space $\mathrm{M}(n,m)$ of $n\times m$ matrices

$$E_{ij}=\sum_{\alpha} x_{i\alpha}\partial_{j\alpha}\,, \tag{1.1}$$

where x_{ij} are the natural coordinates in $\mathrm{M}(n,m)$ and ∂_{ij} are the dual partial derivatives. Suppose $m\geq n$. Identify the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(n))$ with the algebra of differential operators with polynomial coefficients on $\mathrm{M}(n,m)$ invariant under the action of $GL(m)$ by right multiplication.

We obtain an explicit expression for a very large and remarkable family of right-invariant differential operators on $\mathrm{M}(n,m)$ in terms of generators E_{ij}. In particular, our result is a generalization of the following classical Capelli identity [C]

$$\sum_{1\leq i_1<i_2<\dots<i_k\leq n} \operatorname{row-det}\begin{pmatrix} E_{i_1i_1} & E_{i_1i_2} & \dots & E_{i_1i_k}\\ E_{i_2i_1} & E_{i_2i_2}+1 & & E_{i_2i_k}\\ \vdots & & \ddots & \vdots\\ E_{i_ki_1} & E_{i_ki_2} & \dots & E_{i_ki_k}+k-1\end{pmatrix}= \tag{1.2}$$
$$\sum_{1\leq i_1<i_2<\dots<i_k\leq n}\ \sum_{1\leq j_1<j_2<\dots<j_k\leq m} \det(x_{i_aj_b})_{1\leq a,b,\leq k}\ \det(\partial_{j_ai_b})_{1\leq a,b,\leq k}\,,$$

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where $k = 1, 2, \dots, n$ and the row-determinant of a matrix $A = (A_{ij})$ with entries in a non-commutative algebra $\mathbb{A}$ is defined by the following formula

$$\text{row-det}\, A = \sum_{s \in S(n)} \operatorname{sgn}(s)\, A_{1\, s(1)} A_{2\, s(2)} \dots A_{n\, s(n)} \,.$$

A detailed discussion of this famous result of the classical invariant theory can be found in [HU,KS].

1.2. Introduce some notation. It is convenient to use matrices with entries in a non-commutative algebra. Let E, X, and D denote matrices with entries E_{ij}, x_{ij} and ∂_{ij} respectively. Then (1.1) is equivalent to

$$E = X \cdot D' \,,$$

where prime stands for transposition. Introduce also the matrix

$$E - u = (E_{ij} - u \cdot \delta_{ij})_{ij} \,,$$

which depends on a formal parameter u.

A $n \times n$ matrix A with entries A_{ij} in a non-commutative algebra $\mathbb{A}$ can be considered as an element

$$A = \sum_{ij} A_{ij} \otimes e_{ij} \ \in \ \mathbb{A} \otimes \mathrm{M}(n) \,,$$

where e_{ij} are standard matrix units in $\mathrm{M}(n)$. The tensor product of two such matrices A and B is defined by

$$A \otimes B = \sum_{i,j,k,l} A_{ij} B_{kl} \otimes e_{ij} \otimes e_{kl} \ \in \mathbb{A} \otimes \mathrm{M}(n)^{\otimes 2} \,.$$

Define the trace of an element of $\mathbb{A} \otimes \mathrm{M}(n)^{\otimes n}$ by

$$\operatorname{tr} \Big(\sum_{i_1, j_1, \dots, i_n, j_n} A_{i_1, j_1, \dots, i_n, j_n} \otimes e_{i_1, j_1} \otimes \dots \otimes e_{i_n, j_n} \Big) = \sum_{i_1 \dots, i_n} A_{i_1, i_1, \dots, i_n, i_n} \ \in \mathbb{A} \,.$$

The symmetric group $S(k)$ acts in the vector space of k-tensors, so that we have a representation

$$S(k) \to \mathrm{M}(n)^{\otimes k} \,.$$

Let μ be a Young diagram with k boxes. Let V^μ be the corresponding irreducible $S(k)$-module and let χ^μ be its character. Consider χ^μ as an element of the group algebra of $S(k)$

$$\chi^\mu = \sum_{s \in S(k)} \chi^\mu(s) \cdot s \ \in \mathbb{R}[S(k)] \,.$$

Let T be a Young tableau of shape μ and let v_T be the corresponding vector in the Young basis for V^μ. (We recall some basic facts from representation theory of $S(k)$ in section 3 below.) Let T and T' be two Young tableaux of shape μ. Consider the following matrix element

$$\Psi_{TT'} = \sum_{s \in S(k)} (s \cdot v_T, v_{T'}) \cdot s^{-1} \ \in \mathbb{R}[S(k)] \,.$$

Let $\alpha = (i, j)$ be a box from μ. The number

$$c(\alpha) = j - i$$

is called the content of the box α. Write $c_T(i)$ for the content of the box number i in the tableau T.

1.3. It is not difficult to see (see [MNO] or below) that the Capelli identity can be restated as follows

$$\operatorname{tr}\left(E\otimes(E+1)\otimes\cdots\otimes(E+k-1)\cdot\chi^{(1^k)}\right)=\operatorname{tr}\left(X^{\otimes k}\cdot(D')^{\otimes k}\cdot\chi^{(1^k)}\right).$$

The main result we prove in this paper is the following identity

Main Theorem. *Let T and T' be two Young tableaux of the same shape. Then*

$$(E-c_T(1))\otimes\cdots\otimes(E-c_T(k))\cdot\Psi_{TT'}=X^{\otimes k}\cdot(D')^{\otimes k}\cdot\Psi_{TT'}\,. \tag{1.3}$$

The proof is based of some remarkable properties of Jucys-Murphy elements (3.2).

This matrix identity is equivalent to the following n^{2k} identities: for any two k-tuples of indexes

$$i_1,\dots,i_k,\quad j_1,\dots,j_k$$

we have the equality of the corresponding matrix elements

$$\sum_{s\in S(k)}(s\cdot v_T,v_{T'})(E_{i_1j_{s(1)}}-c_T(1)\delta_{i_1j_{s(1)}})\dots(E_{i_kj_{s(k)}}-c_T(k)\delta_{i_kj_{s(k)}})= \tag{1.4}$$
$$\sum_{s\in S(k)}(s\cdot v_T,v_{T'})\sum_{\alpha_1,\dots,\alpha_k}x_{i_1\alpha_1}\dots x_{i_k\alpha_k}\,\partial_{j_{s(1)}\alpha_1}\dots\partial_{j_{s(k)}\alpha_k}\,.$$

In contrast to Capelli identity, these identities involve not only the generators of the center $\mathfrak{Z}(\mathfrak{gl}(n))$ of $\mathcal{U}(\mathfrak{gl}(n))$. It is easy to see that linear combinations of (1.3) span the whole algebra $\mathcal{U}(\mathfrak{gl}(n))$. Moreover, it is easy to see that there are much more identities (1.3) than linearly independent elements of $\mathcal{U}(\mathfrak{gl}(n))$.

Observe also that the identity (1.3) is linear in $v_{T'}$; therefore this vector can be replaced by an arbitrary vector in V^μ.

The matrix

$$\mathbb{E}_{TT'}=(E-c_T(1))\otimes\cdots\otimes(E-c_T(k))\cdot\Psi_{TT'}$$

should be called, perhaps, the *fusion* of k matrices E as its structure is the same as the fusion of R-matrices, see [Ch,KuSR,KuR]. The identity (1.3) is one of the interesting properties of this matrix.

In section 5 we specialize the identities for central elements of $\mathcal{U}(\mathfrak{gl}(n))$ by taking trace. We recover higher Capelli identites introduced in the author's paper [Ok]. We have to mention that the proof of (1.3) given below is direct and does not require R-matrix formalism used in [Ok].

The element

$$\mathbb{S}_\mu=(\dim\mu\,/\,k!\,)\operatorname{tr}\mathbb{E}_{TT}$$

depends only on μ, not on T; it was called in [Ok] the *quantum μ-immanant.* Quantum immanants form a very destinguished linear basis in $\mathfrak{Z}(\mathfrak{gl}(n))$. We recall only some basic facts about them from [Ok] and [OO].

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2. Wick formula.

2.1. Let V be a vector space. By $D(V)$ denote the algebra of differential operators on V with polynomial coefficients. The algebra $D(V)$ is generated by constant vector fields $v \in V$ and by multiplications by linear functions $\xi \in V^*$ subject to Heisenberg commutation relations

$$[v, \xi] = \langle \xi, v \rangle ,$$

where $\langle \, , \, \rangle$ is the canonical pairing

$$V^* \otimes V \to \mathbb{R} .$$

By $S(V^* \oplus V)$ denote the symmetric algebra of the vector space $V^* \oplus V$. Introduce the following linear isomorphism

$$S(V^* \oplus V) \xrightarrow{::} D(V) ,$$

called *normal ordering*. By definition, the normal ordering places all multiplications by functions to the left and all constant vector fields to the right; for example,

$$:\xi_1 v_1 v_2 \xi_2 \xi_3 v_3: = \xi_1 \xi_2 \xi_3 v_1 v_2 v_3 , \qquad \xi_i \in V^*, v_i \in V .$$

By definition, put

$$\underbrace{A\,B} = AB - :AB: , \qquad A, B \in V^* \oplus V . \tag{2.1}$$

This is a number; it is linear in A and B. Clearly,

$$\begin{aligned} \underbrace{v\,\xi} &= \langle \xi, v \rangle , \\ \underbrace{\xi\,v} &= 0 , \qquad \xi \in V^*, v \in V . \end{aligned}$$

Let the pairing

$$\dots \underbrace{A \dots B} \dots$$

mean that this pair should be replaced by the number (2.1). The following theorem can be easily proved by induction.

Theorem (Wick). *Suppose $A_1, \dots, A_k \in V^* \oplus V \subset D(V)$. Then*

$$\begin{aligned} A_1 \dots A_k = {} & :A_1 \dots A_k: + \\ & \sum_{1 \le i < j \le k} :A_1 \dots \underbrace{A_i \dots A_j} \dots A_k: + \\ & \sum_{i,j,p,q} :A_1 \dots \underbrace{A_i \dots A_j} \dots \underbrace{A_p \dots A_q} \dots A_k: + \dots , \end{aligned}$$

where the sum is over all possible pairings in the set $\{1, \dots, k\}$.

For example,

$$A_1 A_2 A_3 = :A_1 A_2 A_3: + :\underbrace{A_1 A_2}\, A_3: + :A_1\, \underbrace{A_2 A_3}: + :\underbrace{A_1 A_2 A_3}: .$$

2.2. Recall that we identify $\mathcal{U}(\mathfrak{gl}(n))$ with the algebra of right-invariant differential operators on $\mathrm{M}(n,m)$ with polynomial coefficients.

Consider the following linear isomorphism

$$S(\mathfrak{gl}(n)) \xrightarrow{\sigma} \mathcal{U}(\mathfrak{gl}(n))$$

introduced by G. Olshanski in [Ol1]; in [KO] it was called the *special* symmetrization. The definition of σ is equivalent to the following (see lemma 2.2.12 in [Ol1])

$$\sigma(E_{i_1j_1}\dots E_{i_kj_k}) = \sum_{\alpha_1,\dots,\alpha_k} x_{i_1\alpha_1}\dots x_{i_k\alpha_k}\partial_{j_1\alpha_1}\dots\partial_{j_k\alpha_k}\,. \tag{2.2}$$

It is easy to see that the RHS of (2.2) is a right-invariant differential operator and hence an element of $\mathcal{U}(\mathfrak{gl}(n))$. By analogy to the normal ordering let us call the map σ the *normal* symmetrization and denote it by colons

$$:E_{i_1j_1}\dots E_{i_kj_k}:=\sigma(E_{i_1j_1}\dots E_{i_kj_k})\,.$$

Suppose $A,B\in\mathfrak{gl}(n)$. Put

$$\underbrace{A\,B} = AB - :AB: \quad \in \mathfrak{gl}(n)\,.$$

It is easy to see that this is simply the matrix multiplication

$$\underbrace{E_{ij}E_{pq}} = \delta_{jp}E_{iq}\,.$$

Observe that chain pairings like

$$:A_1\dots \underbrace{A_a\dots}\underbrace{A_b\dots A_c}\dots A_k:\ , \tag{2.3}$$

where the end of a brace is at the same time the beginning of a new brace, make perfect sence in the case of $\mathfrak{gl}(n)$. The pairing (2.3) simply means that the three matrices should be replaced by their matrix product. The following theorem is lemma 2.2.13 in [Ol1]. We deduce it from the Wick formula.

THEOREM (OLSHANSKI). *Suppose $A_1,\dots,A_k\in\mathfrak{gl}(n)\subset\mathcal{U}(\mathfrak{gl}(n))$. Then*

$$\begin{aligned} A_1\dots A_k =&:A_1\dots A_k: + \\ &\sum_{1\le a<b\le k} :A_1\dots\underbrace{A_a\dots A_b}\dots A_k: + \\ &\sum_{a,b,c}:A_1\dots\underbrace{A_a\dots}\underbrace{A_b\dots A_c}\dots A_k: + \dots\,, \end{aligned}$$

where the sum is over all (possibly chain) pairings in the set $\{1,\dots,k\}$.

For example,

$$A_1A_2A_3 =:A_1A_2A_3: + :A_1A_2\,A_3: + :A_1\,A_2A_3: + :A_1A_2A_3: + :A_1A_2\,A_3:\,.$$

Note that the sum in theorem is in fact the sum over all partitions of the set $\{1,\dots,k\}$ into disjoint union of its subsets (*clusters*)

$$\{i_1,i_2,\dots\},\{j_1,j_2,\dots\},\dots\subset\{1,\dots,k\}.$$

Each cluster $\{i_1,i_2,i_3,\dots\}$ corresponds to the following chain pairing

$$:A_1\dots\underbrace{A_{i_1}\dots}\underbrace{A_{i_2}\dots}\underbrace{A_{i_3}\dots\qquad}\dots A_k:.$$

Proof. Apply the Wick formula to the product

$$E_{i_1j_1}\dots E_{i_kj_k}=\sum_{\alpha_1,\dots,\alpha_k}x_{i_1\alpha_1}\partial_{j_1\alpha_1}\dots x_{i_k\alpha_k}\partial_{j_k\alpha_k}\,. \tag{2.4}$$

Remark that the pairing of $\partial_{j_p\alpha_p}$ with $x_{i_q\alpha_q}, p<q$, induces matrix multiplication of $E_{i_pj_p}$ and $E_{i_qj_q}$. $\square$

3. Young basis.

3.1. Recall the construction of the Young orthogonal basis in the irreducible representations of the symmetric groups $S(k)$, $k=1,2,\dots$. Define it by induction.

The group $S(1)$ is trivial. We can choose any nonzero vector in its unique irreducible representation. Suppose $k>1$. Let $\lambda,|\lambda|=k$, be a Young diagram and let V^λ be the corresponding irreducible $S(k)$-module. Let μ be another Young diagram. Write $\mu\nearrow\lambda$ if $\mu\subset\lambda$ and $|\mu|=|\lambda|-1$. The Young branching rule asserts that

$$V^\lambda=\bigoplus_{\mu\nearrow\lambda}V^\mu\qquad\text{as a } S(k-1)\text{-module}\,. \tag{3.1}$$

Here the sum is orthogonal with respect to the $S(k)$-invariant inner product $(\ ,\ )$ in V^λ. By definition, the Young basis in V^λ is the union of the Young bases in direct summands in (3.1).

It is clear that the Young basis in V^λ is indexed by the following chains of diagrams

$$\varnothing=\lambda^{(0)}\nearrow\lambda^{(1)}\nearrow\dots\nearrow\lambda^{(k-1)}\nearrow\lambda^{(k)}=\lambda\,.$$

Such a chain is the protocol of a Young diagram growth from the empty diagram to the diagram λ. This growth can be also represented as follows: for all $i=1,\dots,k$ put the number i into the box $\lambda^{(i)}/\lambda^{(i-1)}$ of the diagram λ. Then we obtain a *Young tableau* of shape λ, that is a tableau T whose entries strictly increase along each row and down each column. Denote by v_T the Young basis vector corresponding to the tableau T.

By our definition each basis vector is defined only up to a scalar factor. In the sequel we suppose that

$$(v_T,v_T)=1\,.$$

This normalization is the only object in this paper which is not defined over the field $\mathbb{Q}$ of rational numbers.

Suppose $\alpha=(i,j)$ is a box of λ. Recall that the number

$$c(\alpha)=j-i$$

is called the *content* of the box α. For all $i=1,\dots,k$ put

$$c_T(i)=c(\lambda^{(i)}/\lambda^{(i-1)})\,,$$

this is the content of the i-th box in the tableau T. Observe that always

$$c_T(1)=0\,.$$

3.2. For all $i = 1, \dots, k$ consider the following elements of $\mathbb{R}[S(k)]$

$$X_i = (1\,i) + (2\,i) + \cdots + (i-1\ i)\,. \tag{3.2}$$

In particular, $X_1 = 0$. These elements were introduced by Jucys [Ju] and Murphy [Mu]. The following proposition is also due to these authors. Our proof follows [Ol2], section 4.6.

PROPOSITION. *For all* $i = 1, \dots, k$

$$X_i v_T = c_T(i) v_T\,.$$

PROOF. For all $p = 1, \dots, k$ put

$$\Sigma_p = \sum_{1 \le i < j \le p} (i\,j) \quad \in \mathbb{R}[S(p)]\,.$$

It is clear that Σ_p is a central element of $\mathbb{R}[S(p)]$ and it is proved, for example, in [M], Exercise I.7.7, that in all irreducible $S(p)$-modules V^η

$$\Sigma_p|_{V^\eta} = \frac{1}{2} \sum_i \left(\eta_i^2 - (2i-1)\eta_i\right) \cdot \mathrm{id}_{V^\eta}\ . \tag{3.3}$$

Clearly,

$$X_i = \Sigma_i - \Sigma_{i-1}\,. \tag{3.4}$$

Choose q so that $\lambda_q^{(i)} = \lambda_q^{(i-1)} + 1$. Put $l = \lambda_q^{(i)}$. By (3.3) and (3.4) we have

$$\begin{aligned} X_i|_{V^{\lambda^{(i-1)}}} &= \frac{1}{2}(l^2 - (2i-1)l - (l-1)^2 + (2i-1)l) \cdot \mathrm{id}_{V^{\lambda^{(i-1)}}} \\ &= (l-i) \cdot \mathrm{id}_{V^{\lambda^{(i-1)}}} \\ &= c_T(i) \cdot \mathrm{id}_{V^{\lambda^{(i-1)}}}\ . \end{aligned}$$

Since $v_T \in V^{\lambda^{(i-1)}}$ this proves the proposition. $\square$

Let T, T' be two Young tableaux of shape λ. Consider the matrix element

$$\psi_{TT'}(s) = (s \cdot v_T, v_{T'})\,.$$

Consider the following element of $\mathbb{R}[S(k)]$

$$\Psi_{TT'} = \sum_{s \in S(k)} (s \cdot v_T, v_{T'}) \cdot s^{-1}\,.$$

COROLLARY. *For all $i = 1, \dots, k$*

$$X_i \Psi_{TT'} = c_T(i) \Psi_{TT'} \tag{3.5}$$

$$\Psi_{TT'} X_i = c_{T'}(i) \Psi_{TT'} \tag{3.6}$$

PROOF. The equalities (3.5) and (3.6) are equivalent to

$$\sum_{j, j<i} \psi_{TT'}(s(i\,j)) = c_T(i) \psi_{TT'}(s) \tag{3.5'}$$

$$\sum_{j, j<i} \psi_{TT'}((i\,j)s) = c_{T'}(i) \psi_{TT'}(s)\,, \tag{3.6'}$$

which follow from the definition (3.2), the proposition, and the invariance of the inner product

$$(s \cdot v, u) = (v, s^{-1} \cdot u), \quad v, u \in V^\lambda\,. \quad \square$$

It follows from the orthogonality relations for matrix elements and it also follows from the corollary that in the Young basis the operator $\Psi_{TT'}$ is proportional to a matrix unit. Put

$$P_{TT'} = (\dim \mu \,/\, k!)\, \Psi_{TT'}\,,$$

where $\dim \mu$ is the dimension of V^μ. Then

$$\begin{aligned} P_{TT'} \cdot v_{T'} \ &= v_T, \\ P_{TT'} \cdot v_{T''} &= 0, \quad T'' \neq T'\,. \end{aligned}$$

REMARK. The corollary asserts that the matrix elements $\Psi_{TT'}$ form the unique up to scalar factors common eigenbasis for $2k$ commuting self-adjoint operators which act by multiplications by $X_1, \dots, X_k$ from the left and from the right. In fact the representation theory of the symmetric groups can be rediscovered from some simple properties of these operators, see [OV].

3.3. The practical computation of matrix elements $\psi_{TT'}$ is a quite difficult problem. A way of computing them is the following. First one obtains one particular matrix element in each irreducible representation and then the other from this one.

It is known [JK] that in V^λ there is the unique up to scalar factor vector invariant under the action of the group

$$S(\lambda) = S(\lambda_1) \times S(\lambda_2) \times \dots$$

which is the stabilizer of the subsets

$$\{1, \dots, \lambda_1\}, \{\lambda_1 + 1, \dots, \lambda_1 + \lambda_2\}, \{\lambda_1 + \lambda_2 + 1, \dots\}, \dots\,.$$

It can be easily deduced from the definition of the Young basis that this vector is simply the vector v_T, where T is the following Young tableau

$$T = \begin{matrix} 1 & 2 & \dots & \dots & \dots & \lambda_1 - 1 & \lambda_1 \\ \lambda_1 + 1 & \lambda_1 + 2 & \dots & \lambda_1 + \lambda_2 & & & \\ \lambda_1 + \lambda_2 + 1 & \dots & & & & & \end{matrix}\,,$$

which is called the *row tableau* of shape λ. Denote this tableau by T^0. Consider the *Young symmetrizer* [JK] corresponding to the tableau T^0

$$\mathcal{P}\mathcal{Q} \in \mathbb{R}[S(k)]\,,$$

where

$$\mathcal{P} = \sum_{s \in S(\lambda)} s$$

is the row-symmetrizer of the tableau T^0 and

$$\mathcal{Q} = \sum_{s \text{ preserves columns of } T^0} \operatorname{sgn}(s) \cdot s$$

is the column-antisymmetrizer of T^0. Both $\mathcal{P}$ and $\mathcal{Q}$ are up to a scalar factor orthogonal projections

$$\mathcal{P}^* = \mathcal{P}, \quad \mathcal{P}^2 = \lambda!\,\mathcal{P}, \quad \mathcal{Q}^* = \mathcal{Q}, \quad \mathcal{Q}^2 = (\lambda')!\,\mathcal{Q}\,,$$

where $*$ is the involution in $\mathbb{R}[S(k)]$ induced by

$$s \mapsto s^{-1}, \quad s \in S(k)\,,$$

and $\lambda! = \lambda_1!\,\lambda_2!\,\dots$. The element $\mathcal{P}$ acts in V^λ up to a scalar as the orthogonal projection onto v_{T^0}.

It is known that the product $\mathcal{P}\mathcal{Q}$ vanishes in any irreducible representation of $S(k)$ different from V^λ. The operator

$$\mathcal{P}\mathcal{Q}\mathcal{P} = \frac{1}{(\lambda')!}\mathcal{P}\mathcal{Q}(\mathcal{P}\mathcal{Q})^*$$

is a nonzero operator proportional to the orthogonal projection onto v_{T^0} and hence it is proportional to $\Psi_{T^0T^0}$. It can be easily shown that

$$\Psi_{T^0T^0} = \frac{1}{\lambda!}\mathcal{P}\mathcal{Q}\mathcal{P}\,.$$

3.4. Now suppose v is a common eigenvector of the elements $X_1, \dots, X_k$ in a $S(k)$-module V

$$X_i \cdot v = a_i v, \quad i = 1, \dots, k, \quad a_i \in \mathbb{R}\,.$$

There is a standard general method to construct new eigenvectors of $X_1, \dots, X_k$ from v. Put

$$s_i = (i\,, i+1), \quad i = 1, \dots, k-1\,.$$

It is easy to check [Mu] that

$$s_i X_i + 1 = X_{i+1} s_i\,, \tag{3.7}$$

$$s_i X_j = X_j s_i\,, \quad j \neq i, i+1\,. \tag{3.8}$$

Suppose

$$a_{p+1} \neq a_p \pm 1 \tag{3.9}$$

for some p. Put

$$v' = \left(s_p - \frac{1}{a_{p+1} - a_p} \right) \cdot v \quad \in V\,.$$

By (3.9) we have $v' \neq 0$. It follows from (3.7) and (3.8) that

$$X_i \cdot v' = a_{s_p(i)} v'\,.$$

It is easy to see that all eigenvectors $\Psi_{TT'}$ in the $S(k) \times S(k)$-module $\mathbb{R}[S(k)]$ can be obtained in this way from an arbitrary initial matrix element (for example, $\Psi_{T^0T^0}$) in each irreducible representation.

4. Proof of the main theorem

4.1. We have to prove the matrix equality (1.3). Prove that all matrix elements are equal. Put $\psi(s) = \psi_{TT'}(s)$ and put $c(i) = c_T(i)$. By (2.2) we have to prove that for all collections of indexes

$$i_1, \dots, i_k, \quad j_1, \dots, j_k$$

we have

$$\sum_{s \in S(k)} \psi(s) \cdot \left(E_{i_1 j_{s(1)}} - c(1)\delta_{i_1 j_{s(1)}}\right) \dots \left(E_{i_k j_{s(k)}} - c(k)\delta_{i_k j_{s(k)}}\right) \\ = \sum_{s \in S(k)} : \psi(s) E_{i_1 j_{s(1)}} \dots E_{i_k j_{s(k)}} : \quad . \tag{4.1}$$

To simplify notation put

$$l_p = j_{s(p)}\,, \quad p = 1, \dots, k\,.$$

The indexes $l_1, \dots, l_k$ vary simultaneously with the permutation $s \in S(k)$. We are going expand out all brackets in the LHS of (4.1) and then apply the theorem from paragraph 2.2 to all monomials in E_{ij}.

Fix some s to see what happens. We have the product

$$(E_{i_1 l_1} - c(1)\delta_{i_1 l_1}) \dots (E_{i_k l_k} - c(k)\delta_{i_k l_k})\;.$$

First for all $p = 1, \dots, k$ we have to choose in the p-th bracket either $E_{i_p l_p}$ or $(-c(p)\delta_{i_p l_p})$. Let us depict our choice as a diagram like

$$\begin{array}{cccccc} {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & & {\scriptstyle k-1} & {\scriptstyle k} \\ \circ & * & \circ & \dots & \circ & * \end{array} \;,$$

where the circles represent the factors $E_{i_p l_p}$ and the asterisks represent the factors $(-c(p)\delta_{i_p l_p})$. For example, the diagram

$$\begin{array}{cccccc} {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & & {\scriptstyle k-1} & {\scriptstyle k} \\ \circ & \circ & \circ & \dots & \circ & \circ \end{array} \tag{4.2}$$

corresponds to the product

$$E_{i_1 l_1} \dots E_{i_k l_k}\,,$$

and the diagram

$$\begin{array}{cccccc} {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & & {\scriptstyle k-1} & {\scriptstyle k} \\ * & * & * & \dots & * & * \end{array}$$

corresponds to

$$(-c(1)\delta_{i_1 l_1}) \dots (-c(k)\delta_{i_k l_k})\;.$$

Next, we have to divide the factors $E_{i_p l_p}$ (or, equivalently, the circles in the diagram) into clusters in all possible ways. This will be depicted as follows: a cluster $\{a, b, c\}$

$$\begin{array}{cccccccc} & {\scriptstyle a} & & & {\scriptstyle b} & & {\scriptstyle c} & \\ \circ & \circ & * & \circ & \circ & \circ & \circ & * \end{array}$$

corresponds to the factor

$$:\ldots\delta_{l_a i_b}\delta_{l_b i_c}E_{i_a l_c}\ldots: \quad .$$

We see that the summands which arise in the LHS of (4.1) are indexed by permutations s and diagrams like

$$\underbrace{\overset{1}{\circ}\qquad\overset{2}{\circ}}\qquad \overset{3}{*}\qquad \cdots \quad .$$

Denote the corresponding summand by

$$\left[\quad s \quad\middle|\quad \underbrace{\overset{1}{\circ}\qquad\overset{2}{\circ}}\qquad \overset{3}{*}\qquad \cdots \quad\right] \quad .$$

In order to establish (4.1) we have to show that all summands cancel each other except those corresponding to the trivial diagram (4.2).

4.2. To explain the idea of the proof, we show first that all summands that contain exactly $k-1$ factors E_{ij} cancel. Such summands correspond to two kind of diagrams:

$$\circ \quad \cdots \quad \circ \qquad \overset{b}{*} \qquad \circ \quad \cdots \quad \circ \qquad b = 1, \ldots, k\,, \tag{4.3}$$

and

$$\circ \quad \cdots \quad \underbrace{\overset{a}{\circ} \quad \cdots \quad \overset{b}{\circ}} \quad \cdots \quad \circ \qquad 1 \le a < b \le k\,. \tag{4.4}$$

We claim that for all s and all b

$$\left[\quad s \quad\middle|\; \cdots \quad \overset{b}{*} \quad \cdots \right] + \sum_{a<b} \left[s(ab) \middle|\; \cdots \quad \underbrace{\overset{a}{\circ} \quad \cdots \quad \overset{b}{\circ}} \quad \cdots \right] = 0\,. \tag{4.5}$$

In fact, all summands in (4.5) are proportional to

$$:\delta_{i_b l_b}\prod_{p\ne b}E_{i_p l_p}: \quad ,$$

and the coefficient equals

$$-c(b)\psi(s) + \sum_{a,a<b}\psi(s(ab))\,. \tag{4.6}$$

By (3.5') this number equals zero. Evidently, by (4.5) all summands with diagrams (4.3) and (4.4) cancel.

4.3. Now consider the general case. Suppose we have a diagram Γ, for example

$$\Gamma = \underbrace{\overset{1}{\circ}\qquad\overset{2}{\circ}\qquad\overset{3}{*}\qquad\underbrace{\overset{4}{\circ}\qquad\overset{5}{*}\qquad\overset{6}{\circ}}\qquad\overset{7}{\circ}} \quad .$$

This diagram corresponds to three clusters

$$\{1, 7\}, \{2\}, \{4, 6\}$$

and the subset

$$\{3,5\}$$

formed by all asterisks. Denote this asterisk subset by As.

Let b be the smallest positive integer such that b is not a beginning of a new circle cluster. In our example $b=3$. We say that the diagram Γ is of the *first kind* if $b \in As$ and of the *second kind* otherwise. For example, our diagram in example and all diagrams (4.3) are of the first kind, whereas all diagrams (4.4) are of the second kind.

We claim that for all s and for all diagrams Γ of the first kind the corresponding summand

$$\left[\; s \;\middle|\; \cdots \quad \overset{b}{*} \quad \cdots \;\right] \tag{4.7}$$

cancels with the sum

$$\sum_{a<b} \left[\; s(ab) \;\middle|\; \cdots \quad \underbrace{\overset{a}{\circ} \quad \cdots \quad \overset{b}{\circ}} \quad \cdots \;\right], \tag{4.8}$$

where the pairing of a and b means that b should be added to the cluster that begins with a. Note that the diagrams in (4.8) are of the second kind and all summands with a second kind diagram appear exactly one time in the sum (4.8) while s ranges over $S(k)$ and Γ ranges over all diagrams of the first kind.

In our example

$$\begin{aligned}
&\left[\; s \;\middle|\; \underbrace{\overset{1}{\circ} \quad \overset{2}{\circ} \quad \overset{3}{*} \quad \underbrace{\overset{4}{\circ} \quad \overset{5}{*} \quad \overset{6}{\circ}} \quad \overset{7}{\circ}} \;\right] + \\
&\left[\; s(1\,3) \;\middle|\; \underbrace{\overset{1}{\circ} \quad \overset{2}{\circ} \quad} \underbrace{\overset{3}{\circ} \quad \underbrace{\overset{4}{\circ} \quad \overset{5}{*} \quad \overset{6}{\circ}} \quad \overset{7}{\circ}} \;\right] + \\
&\left[\; s(2\,3) \;\middle|\; \underbrace{\overset{1}{\circ} \quad \underbrace{\overset{2}{\circ} \quad \overset{3}{\circ}} \quad \underbrace{\overset{4}{\circ} \quad \overset{5}{*} \quad \overset{6}{\circ}} \quad \overset{7}{\circ}} \;\right] = 0
\end{aligned}$$

The cancellation of (4.7) and (4.8) is proved in the same way as (4.5). It is easy to see that all summands in (4.7) and (4.8) are proportional. Indeed, suppose $\{a,c,d,\dots,z\}$ is the cluster in Γ that begins with a.

$$\cdots \underbrace{\overset{a}{\circ} \cdots \overset{b}{*} \cdots \overset{c}{\circ}}\underbrace{\cdots \overset{d}{\circ}}\underbrace{\cdots} \;\cdots\; \underbrace{\cdots \overset{z}{\circ}} \cdots$$

Then the contribution of this cluster to (4.7) is the following factor

$$\delta_{l_a i_c} \delta_{l_c i_d} \dots E_{i_a l_z}\,.$$

The contribution of the asterisk on the b-th place is the factor

$$-c(b)\delta_{l_b i_b}\,.$$

On the other hand, the contribuition of the cluster $\{a,b,c,d,\dots,z\}$

$$\cdots \underbrace{\overset{a}{\circ} \cdots \overset{b}{\circ}}\underbrace{\cdots \overset{c}{\circ}}\underbrace{\cdots \overset{d}{\circ}}\underbrace{\cdots} \;\cdots\; \underbrace{\cdots \overset{z}{\circ}} \cdots$$

to the a-th summand in (4.8) is the factor

$$\delta_{l_b i_b} \delta_{l_a i_c} \delta_{l_c i_d} \dots E_{i_a l_z}\,.$$

Therefore all summands in (4.7) and (4.8) are proportional. The coefficient equals (4.6) again and hence equals zero. This concludes the proof of the theorem.

5. Quantum immanants and higher Capelli identities.

5.1. In this section we specialize the main theorem for central elements of $\mathcal{U}(\mathfrak{gl}(n))$. Recall that the trace of an element of $\mathcal{U}(\mathfrak{gl}(n)) \otimes \mathrm{M}(n)^{\otimes n}$ is defined by

$$\operatorname{tr}\Big(\sum_{i_1,j_1,\dots,i_n,j_n} A_{i_1,j_1,\dots,i_n,j_n} \otimes e_{i_1,j_1} \otimes \dots \otimes e_{i_n,j_n} \Big) = \sum_{i_1\dots,i_n} A_{i_1,i_1,\dots,i_n,i_n} \in \mathcal{U}(\mathfrak{gl}(n))\,.$$

Let $\mu, |\mu| = k$ be a Young diagram. Denote by $\operatorname{Tab}(\mu)$ the set of all Young tableaux of shape μ. Put

$$\dim \mu = \dim V^\mu\,.$$

Recall that we consider the character χ^μ of the module V^μ as an element of $\mathbb{R}[S(k)]$

$$\chi^\mu = \sum_{s \in S(k)} \chi^\mu(s) \cdot s \in \mathbb{R}[S(k)]\,.$$

Put

$$\mathbb{E}_T = \mathbb{E}_{TT}\,,$$

where the fusion matrix $\mathbb{E}_{TT'}$ is defined by

$$\mathbb{E}_{TT'} = (E - c_T(1)) \otimes \dots \otimes (E - c_T(k)) \cdot \Psi_{TT'}\,.$$

The following theorem is the main result of [Ok]. Here we deduce its first claim from the main theorem.

Theorem [Ok].

(a) *For all* $T \in \operatorname{Tab}(\mu)$

$$\operatorname{tr} \mathbb{E}_T = \frac{1}{\dim \mu} \operatorname{tr} X^{\otimes k} (D')^{\otimes k} \chi^\mu \quad \in \mathcal{U}(\mathfrak{gl}(n))\,. \tag{5.1}$$

In particular, the LHS of (5.1) *does not depend on the choice of* $T \in \operatorname{Tab}(\mu)$.

(b) *The element* (5.1) *lies in the center* $\mathfrak{Z}(\mathfrak{gl}(n))$ *of* $\mathcal{U}(\mathfrak{gl}(n))$.

(c) *The elements* (5.1) *form a linear basis of* $\mathfrak{Z}(\mathfrak{gl}(n))$ *indexed by all Young diagrams* μ.

Proof. Prove (a). By the main theorem we have

$$\operatorname{tr} \mathbb{E}_T = \operatorname{tr} X^{\otimes k} (D')^{\otimes k} \Psi_T\,.$$

Since the entries of the matrix X commute we have

$$s \cdot X^{\otimes k} = X^{\otimes k} \cdot s\,, \tag{5.2}$$

for all $s \in S(k)$, and similarly

$$s \cdot (D')^{\otimes k} = (D')^{\otimes k} \cdot s\,. \tag{5.3}$$

Observe that

$$1 / k! \sum s \Psi_T s^{-1} = \frac{1}{\dim \mu} \chi^\mu\,.$$

Therefore

$$\begin{aligned}\operatorname{tr}\mathbb{E}_T &= 1\,/\,k!\sum_{s\in S(k)} \operatorname{tr} s\, X^{\otimes k}(D')^{\otimes k}\Psi_T\, s^{-1}\\ &= \frac{1}{\dim\mu}\operatorname{tr} X^{\otimes k}(D')^{\otimes k}\chi^{\mu}\,.\end{aligned}$$

Prove (b). Prove, for example, that the LHS of (5.1) is a central element. Denote by g_{ij} and $(g^{-1})_{ij}$ the matrix elements of a matrix $g\in GL(n)$ and its inverse matrix g^{-1}. The following equality is obvious

$$\sum_k g_{ki}(g^{-1})_{jk} = \delta_{ij}\,. \tag{5.4}$$

Consider the adjoint action $\operatorname{Ad}(g)$ of g in $\mathfrak{gl}(n)$

$$\operatorname{Ad}(g)\cdot E_{ij} = \sum_{k,l} g_{ki}(g^{-1})_{jl}E_{kl}\,. \tag{5.5}$$

Under the adjoint action of g the entries of the matrix $(E-u)$ are transformed as follows

$$\begin{aligned}(E-u) &\xrightarrow{\operatorname{Ad}(g)} \sum_{i,j}\left(\sum_{k,l} g_{ki}(g^{-1})_{jl}E_{kl}\right)\otimes e_{ij} - u\sum_i 1\otimes e_{ii} &&\text{by (5.5)}\\ &= \sum_{k,l}(E_{kl}-u\delta_{kl})\otimes\left(\sum_{i,j} g_{ki}(g^{-1})_{jl}e_{ij}\right) &&\text{by (5.4)}\\ &= g'(E-u)(g')^{-1} \end{aligned} \tag{5.6}$$

The product (5.5) is the product of the matrix $(E-u)$ with entries in $\mathcal{U}(\mathfrak{gl}(n))$ and two matrices with entries in the ground field. Consider the following element of $\mathcal{U}(\mathfrak{gl}(n))$

$$\operatorname{tr}((E-u_1)\otimes\cdots\otimes(E-u_k)\cdot s)\,, \tag{5.7}$$

where the numbers u_i and the permutation $s\in S(k)$ are arbitrary. By (5.4) the adjoint action of g' takes this element of $\mathcal{U}(\mathfrak{gl}(n))$ to

$$\operatorname{tr}(g^{\otimes k}(E-u_1)\otimes\cdots\otimes(E-u_k)(g^{-1})^{\otimes k}\cdot s) = \operatorname{tr}((E-u_1)\otimes\cdots\otimes(E-u_k)\cdot s)\,.$$

Hence,(5.7) is an element of $\mathfrak{Z}(\mathfrak{gl}(n))$. Therefore (5.1) is an element of $\mathfrak{Z}(\mathfrak{gl}(n))$.

Prove (c). Consider the standard filtration in $\mathcal{U}(\mathfrak{gl}(n))$ and consider the isomorphism

$$\operatorname{gr}\mathcal{U}(\mathfrak{gl}(n)) \cong S(\mathfrak{gl}(n))\,.$$

It is clear that

$$\operatorname{tr}\mathbb{E}_T = \frac{1}{\dim\mu}\operatorname{tr} E^{\otimes k}\chi^{\mu} + \text{ lower terms }\,.$$

Suppose $G = (g_{ij})$ is a $n \times n$-matrix. It follows from the classical decomposition of the vector space of tensors that the following polynomial in g_{ij}

$$\operatorname{tr}\, G^{\otimes k} \chi^{\mu} \,/\, k! \tag{5.8}$$

equals the trace of G in the irreducible $GL(n)$-module with highest weight μ (or, equivalently, it equals the Schur polynomial s_μ in the eigenvalues of G). The polynomials (5.8) form a linear basis in the vector space of invariants for the adjoint action of $GL(n)$ on $\mathfrak{gl}(n)$. Hence the elements (5.1) form a linear basis in $\mathfrak{Z}(\mathfrak{gl}(n))$. □

Remark. Given a matrix $A = (a_{ij})$, $i, j = 1, \dots, k$, the number

$$\sum_{s \in S(k)} \chi^{\mu}(s)\, a_{1,s(1)} \dots a_{k,s(k)}$$

is called the *μ-immanant* of the matrix A. If $\mu = (1^k), (k)$ then the μ-immanant turns into determinant and permanent respectively. Observe that (5.8) is the sum of μ-immanants of principal k-submatrices (with repeated rows and columns) of the matrix G.

5.2. By definition, put

$$\mathbb{S}_\mu = \frac{\dim \mu}{k!} \operatorname{tr} \mathbb{E}_T\,, \quad T \in \operatorname{Tab}(\mu)\,. \tag{5.9}$$

By the theorem this central element does not depend on the choice of $T \in \operatorname{Tab}(\mu)$. If $\mu = (1^k)$ then the definition of $\mathbb{S}_\mu$ turns into the definition of quantum determinant for $\mathcal{U}(\mathfrak{gl}(n))$ (see [KuS] or [MNO]). By analogy to quantum determinant and because of the structure of the highest term of (5.1) we call $\mathbb{S}_\mu$ the *quantum μ-immanant.* Quantum immanants were introduced and studied in the authors paper [Ok]; from a different point of view they were studied in [OO]. Here we mention some most important properties of these remarkable basis elements of $\mathfrak{Z}(\mathfrak{gl}(n))$.

5.3. We claim that the identity (5.1) is a direct generalization of the classical Capelli identity (1.2). If $\mu = (1^k)$ then it is easy to see that the RHS of (5.1) turns into the RHS of (1.2). Let us show that the LHS of (5.1) turns into the LHS of (1.2). Let

$$i_1, i_2, \dots, i_k$$

be a k-tuple of indexes. Denote by $\iota!$ the order of the stabilizer of this collection in the symmetric group $S(k)$. For example, if all i_j are distinct then $\iota! = 1$.

Teorem [Ok].

$$\begin{aligned}\mathbb{S}_\mu &= \sum_{i_1 \ge \dots \ge i_k} 1/\iota! \sum_{T \in \operatorname{Tab}(\mu)} \sum_{s \in S(k)} \psi_T(s)\,(E_{i_1 i_{s(1)}})(E_{i_2 i_{s(2)}} - c_T(2)\delta_{i_2 i_{s(2)}}) \dots \\ &= \sum_{i_1 \le \dots \le i_k} 1/\iota! \sum_{T \in \operatorname{Tab}(\mu)} \sum_{s \in S(k)} \psi_T(s)\,(E_{i_1 i_{s(1)}})(E_{i_2 i_{s(2)}} - c_T(2)\delta_{i_2 i_{s(2)}}) \dots\end{aligned}$$

In [Ok] this theorem was used in proof of the identity (5.1). Here we deduce this theorem from (5.1).

PROOF. By (5.1) we have

$$\begin{aligned}\mathbb{S}_\mu &= 1\,/\,k! \sum_{T\in\mathrm{Tab}(\mu)} \operatorname{tr}\mathbb{E}_T \\ &= 1\,/\,k! \sum_{i_1,\dots,i_k}\sum_{T\in\mathrm{Tab}(\mu)}\sum_{s\in S(k)} \psi_T(s)\,(E_{i_1 i_{s(1)}})(E_{i_2 i_{s(2)}} - c_T(2)\delta_{i_2 i_{s(2)}})\dots\end{aligned} \tag{5.10}$$

By (5.2) and (5.3) the matrix

$$\sum_{T\in\mathrm{Tab}(\mu)} \mathbb{E}_T = \operatorname{tr} X^{\otimes k}(D')^{\otimes k}\chi^\mu$$

is invariant under conjugation by element of the group $S(k)$. Hence all $k!\,/\,\iota!$ different rearrangement of the indexes $i_1,\dots,i_k$ make the same contribution to the trace (5.10) and hence we can choose an arbitrary (for example, increasing or decreasing) ordering of the indexes. This proves the theorem. □

It is easy to see that the second formula for $\mathbb{S}_\mu$ in the theorem turns into the LHS of (1.2) when $\mu = (1^k)$. Therefore we call the equalities (5.1) the *higher Capelli identities*.

REMARK. The aguments based on the two trivial observations (5.2) and (5.3) and on the main theorem provide an elementary proof of many identities involving the matrix $\mathbb{E}_T$ which seemed to require deep machinery of Yangians and R-matrixes. One of them is the following identity

$$\begin{aligned}P_{T^1T^2}\mathbb{E}_{T^3T^4} &= P_{T^1T^2}X^{\otimes k}(D')^{\otimes k}\Psi_{T^3T^4} \\ &= X^{\otimes k}(D')^{\otimes k}P_{T^1T^2}\Psi_{T^3T^4} \\ &= X^{\otimes k}(D')^{\otimes k}\delta_{T^2T^3}\Psi_{T^1T^4} \\ &= \delta_{T^2T^3}\,\mathbb{E}_{T^1T^4}\,,\end{aligned}$$

where $T_1,\dots,T_4$ are four arbitrary Young tableaux.

5.4. Denote by π_λ the representation of $GL(n)$ with highest weight λ. Recall the definition of the *shifted Schur function* from [OO]. Put

$$(x \downharpoonright k) = x(x-1)\dots(x-k+1),.$$

This product is called falling *factorial power*. Put also

$$\rho = (n-1,\dots,1,0)$$

By definition

$$s^*_\mu(x_1,\dots,x_n) = \frac{\det\left[(x_i+\rho_i \downharpoonright \mu_j+\rho_j)\right]}{\det\left[(x_i+\rho_i \downharpoonright \rho_j)\right]}\,.$$

The relation between quantum immanants and shifted Schur functions is the fol-

THEOREM [Ok]. *Put* $s^*_\mu(\lambda) = s^*_\mu(\lambda_1, \lambda_2, \dots)$. *Then*

$$\pi_\lambda(\mathbb{S}_\mu) = s^*_\mu(\lambda)\,. \tag{5.11}$$

Shifted Schur function have many remarkable properties [OO] (see also [Ok]). Most of these properties have a natural interpretation in terms of quantum immanants $\mathbb{S}_\mu$ and are closely related to higher Capelli identities. One of the main technical tools to handle shifted Schur functions is the following theorem (we shall need it in the proof of (5.11)). An argument very close to our proof was used by S. Sahi in [S]. Shifted Schur function are a particular case of certain remarkable polynomials, which existence was proved in [S]. This particular case is much more simple and can be studied much deeper than the general case considered in [S].

By $\Lambda^*(n)$ denote the algebra of polynomials in variables $x_1, \dots, x_n$ which are symmetric in new variables $x_1 + \rho_1, \dots, x_n + \rho_n$. Such polynomials are called *shifted symmetric* [OO]. It is clear that $s^*_\mu \in \Lambda^*(n)$. Denote by $H(\mu)$ the product of the hook lenghts of all boxes of μ.

CHARACTERIZATION THEOREM [Ok]. *Any of the two following properties determines the polynomial* $s^*_\mu \in \Lambda^*(n)$ *uniquely:*

(A) $\deg s^*_\mu \le |\mu|$ *and*

$$s^*_\mu(\lambda) = \delta_{\mu\lambda} H(\mu)$$

for all λ *such that* $|\lambda| \le |\mu|$;

(B) *the highest term of* s^*_μ *is the ordinary Schur function* s_μ *and*

$$s^*_\mu(\lambda) = 0$$

for all λ *such that* $|\lambda| < |\mu|$.

PROOF OF (5.11). It is well known that the eighenvalue in the representation π_λ of any element of $\mathfrak{Z}(\mathfrak{gl}(n))$ is a shifted symmetric function in λ. Apply $\mathbb{S}_\mu$ to the highest vector v in the representation π_λ. We have

$$\begin{aligned} E_{ii} \cdot v &= \lambda_i v\,, & i &= 1, \dots, n \\ E_{ij} \cdot v &= 0\,, & i &< j\,. \end{aligned}$$

By arguments used in proof of part (c) of the theorem in section 5.1

$$\pi_\lambda(\mathbb{S}_\mu) = s_\mu(\lambda) + \text{ lower terms }.$$

On the other hand it is clear that $\mathbb{S}_\mu$ vanishes in all representations π_λ such that $|\lambda| < |\mu|$. Indeed, these reprentations arise as subrepresentations of the representation of $\mathcal{U}(\mathfrak{gl}(n))$ in the vector space of polynomials on $\mathrm{M}(n,m)$ of degree $|\lambda|$. Such polynomials are clearly annihilated by the differential operator in the RHS of (5.1). Now (5.11) follows from the characterization theorem. $\square$

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